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Approximation by Müntz-Polynomials Away from the Origin

JOSEPH BAK

Department of Mathematics, City College, New York, New York 10031

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Suppose $0 \leqslant a < b$; $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n$. Let $\Lambda = \{1, x^{\lambda_1}, ..., x^{\lambda_n}\}$,

$$\begin{aligned} S &= \text{the Lipschitz class } \mathrm{Lip}_1\ 1[a, b] \\ &= \{f \in C[a, b] : |f(x) - f(y)| \leqslant |x - y| \text{ for } x, y \in [a, b]\}. \end{aligned}$$

The approximation index $I_\Lambda[a, b]$ is defined by

$$I_\Lambda[a, b] = \sup_{f \in S} \inf_{P \in [\Lambda]} \|f - P\|,$$

where $\|\cdot\|$ denotes the sup-norm on $[a, b]$. The importance of $I_\Lambda[a, b]$ as a measure of the closeness of $[\Lambda]$ to arbitrary functions in $C[a, b]$ is discussed, e.g., in [1, p. 440].

$I_\Lambda[0, 1]$ has been determined (to within positive constant factors independent of n) for all sequences $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n$ and takes the special forms

(A) If $\lambda_{j+1} - \lambda_j \leqslant 2$ for $0 \leqslant j \leqslant n - 1$, then

$$I_\Lambda[0, 1] \doteq \left(\sum_{j=1}^{n} \lambda_j\right)^{-1/2}.$$

(B) If $\lambda_{j+1} - \lambda_j \geqslant 2$ for $0 \leqslant j \leqslant n - 1$, then

$$I_\Lambda[0, 1] \doteq \exp\left(-2 \sum_{j=1}^{n} \frac{1}{\lambda_j}\right),$$

where $\doteq$ means "equal up to a constant factor." See [9]. The results in all cases reflect the Müntz condition that the linear span of the *infinite* sequence $\{x^{\lambda_j}\}_{j=0}^{\infty}$ is dense in $C[0, 1]$ iff $\sum_{j=1}^{\infty} (1/\lambda_j)$ diverges.

Regarding the density of the linear span of $\{x^{\lambda_j}\}_{j=0}^{\infty}$ in $C[a, b]$, $a > 0$, it

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was proven by Clarkson and Erdös [4, p. 9] for subsequences of the positive integers and by Luxemburg and Korevaar [8, p. 30] for arbitrary positive sequences $\{\lambda_j\}_{j=0}^{\infty}$ that an identical Müntz condition holds. That is, $[\{x^{\lambda_j}\}_{j=0}^{\infty}]$ is dense in $C[a, b]$ iff $\sum_{j=1}^{\infty} (1/\lambda_j)$ diverges.

Moreover, von Golitschek [6] showed that if, for some $\beta > 0$, $\lambda_j \leqslant \beta j$, for all j, $I_\Lambda[a, 1] \leqslant K_{a,\beta}/n$. In Theorem 1, we refine this result, obtaining an upper bound for the constant $K_{a,\beta}$ with several interesting ramifications.

THEOREM 1. *If* $\lambda_j \leqslant \beta j$, $\beta > 2$, *for all* j

$$I_\Lambda[a, 1] \leqslant \frac{90\gamma}{n}, \qquad \textit{where} \qquad \gamma = \left(\frac{1}{a}\right)^{\beta/2-1} \operatorname{Max}\left\{1, \frac{1}{\log(\beta-1)}\right\}.$$

Proof. Let $d(f; \Lambda)$ denote the uniform distance of f to $[\Lambda]$. Then

$$d(x^k; \Lambda) \leqslant \frac{1}{a^N} \prod_{j=0}^{n} \left|\frac{\lambda_j - k}{\lambda_j + k + 2N}\right|. \tag{1}$$

See [6, p. 22]. (Another proof of (1), indicating the connection with analytic-function theory, can be given as follows: Note that $d(x^k; \Lambda) = \sup \int_a^1 x^k \, du(x)$, where the sup is taken over all measures du of mass 1, orthogonal to Λ. For any such du, let $F(z) = \int_a^1 x^z \, du(x)$. Then F is entire and inequality (1) follows by applying the usual Blaschke estimates to $F(z)$ in the half-plane $\operatorname{Re} z \geqslant -N$.)

Now assume $\lambda_j \leqslant \beta j$, $\beta > 2$, and set $N = (\beta/2 - 1)k$. Suppose moreover that q is such that $\lambda_q \leqslant k < \lambda_{q+1}$. Then we can factor

$$\prod_{j=0}^{n} \left|\frac{\lambda_j - k}{\lambda_j + k + 2N}\right|$$

into

$$P_1 P_2 = \prod_{j=1}^{q} \frac{k - \lambda_j}{k + \lambda_j + 2N} \prod_{j=q+1}^{n} \frac{\lambda_j - k}{\lambda_j + k + 2N}.$$

P_1 is easily seen to be bounded by $(1/(\beta - 1))^q$. To estimate P_2, note that

$$\frac{\lambda_j - k}{\lambda_j + k + 2N} = \frac{\lambda_j + N - (N + k)}{\lambda_j + N + (N + k)}$$

and using the fact that $(1 - u)/(1 + u) \leqslant e^{-2u}$

$$P_2 \leqslant \exp\left[-2(N + k) \sum_{j=q+1}^{n} \frac{1}{\lambda_j + N}\right] \leqslant \exp\left[-\beta k \sum_{j=q+1} \frac{1}{\lambda_j + N}\right].$$

Finally note

$$\sum_{j=q+1}^{n} \frac{1}{\lambda_j + N} \geqslant \sum_{j=q+1}^{n} \frac{1}{\beta j + N} \geqslant \int_{q+1}^{n} \frac{dx}{\beta x + N}$$
$$= \frac{1}{\beta} \log \left(\frac{\beta n + N}{\beta q + \beta + N}\right) \geqslant \frac{1}{\beta} \log \left(\frac{n}{q + 1 + (1/2)k}\right)$$

so that

$$P_2 \leqslant \left(\frac{q + 1 + (1/2)k}{n}\right)^k.$$

We now consider two cases.

Case 1. If $q + 1 \leqslant k$, $P_1 P_2 \leqslant (1/(\beta - 1)^q)(3k/2n)^k \leqslant (3k/2n)^k$ since $\beta > 2$.

Case 2. If $q + 1 > k$, $P_1 P_2 \leqslant (1/(\beta - 1)^q)(5q/2n)^k$ which has its *maximum* for fixed k at $q = k/\log(\beta - 1)$. Hence $P_1 P_2 \leqslant (5k/2ne \log(\beta - 1))^k$.

In either case, we conclude from (1) and the above that

$$d(x^k; \Lambda) \leqslant \left(\frac{3\gamma k}{2n}\right)^k, \quad \text{where} \quad \gamma = \left(\frac{1}{a}\right)^{\beta/2 - 1} \operatorname{Max} \left\{1, \frac{1}{\log(\beta - 1)}\right\}. \tag{2}$$

To complete the proof, let $f \in S$. Note, as in [5, 6] that we can find an ordinary Mth degree polynomial $P_M(x) = \sum_{k=0}^{M} c_k x^k$ such that

$$\|f - P_M\| \leqslant \frac{8}{M}; \quad c_0 = f(a); \quad |c_k| \leqslant \frac{2M^{k-1}}{k!}, \quad k = 1, 2,..., M. \tag{3}$$

Now let $P_\Lambda(x) = \sum_{k=0}^{M} c_k Q_k(x)$, where $Q_k \in [\Lambda]$ is the best Λ-approximator to x^k. Then by (2) and (3), $\|P_M - P_\Lambda\| \leqslant \sum_{k=1}^{M} (2M^{k-1}/k!)(3\gamma k/2n)^k$ and using the fact that $k! > k^k/e^k$

$$\|P_M - P_\Lambda\| \leqslant \frac{2}{M} \sum_{k=0}^{M} \left(\frac{3Me\gamma}{2n}\right)^k. \tag{4}$$

Choosing $M = [n/3e\gamma]$ it follows from (3) and (4) that

$$\|f - P_\Lambda\| \leqslant \|f - P_M\| + \|P_M - P_\Lambda\| \leqslant \frac{90\gamma}{n}$$

and the proof is complete.

Remarks. (1) If $\lambda_j \leqslant 2j$ for all j we can choose $\beta = 2 - 2/\log a$ thus obtaining a minimum value of $\gamma \leqslant e \log(1/a)$ if $a < 1/e$. In particular, $I_\Lambda[1/n^2, 1] \leqslant (A \log n)/n$. The latter inequality has implications for the degree of *rational* approximation on $[0, 1]$. See [3].

(2) Theorem 1 can also be used to show the existence of a finite sequence $\lambda_1(n), \lambda_2(n),..., \lambda_n(n)$ for which $I_\Lambda[0, 1] \geqslant A > 0$ (where A is independent of n) while $I_\Lambda[a, 1] \to 0$ as $n \to \infty$. For, setting $\lambda_1 = \log n$, $\lambda_2 = 2 \log n,..., \lambda_n = n \log n$, it follows that

$$I_\Lambda[0, 1] \geqslant A_1 \exp\left(\frac{-2}{\log n} \sum_{j=1}^{n} \frac{1}{j}\right) \geqslant A_2,$$

see [1], while (taking $\beta = \log n$)

$$I_\Lambda[a, 1] \leqslant A \left(\frac{1}{a}\right)^{(1/2)\log n} \Big/ n = A/n^{1+\log\sqrt{a}}$$

which approaches 0 as $n \to \infty$ as long as $a > 1/e^2$.

(3) The inclusion of the constant 1 in the sequence Λ simplified the proof of Theorem 1 but is actually unnecessary as long as we assume some upper bound for $c_0 = f(a)$. For then, as we shall see below, the constant 1 can be reapproximated by a linear combination of $\{x^{\lambda_j}\}_{j=1}^n$. P. Erdös suggested moreover that it might be interesting to estimate the degree of approximation possible by Λ-polynomials on $[a, 1]$ to x^k for any fixed $k \geqslant 0$. On $[0, 1]$ the distance $d_{[0,1]}(x; \Lambda)$ in many cases yields the lower bound for $I_\Lambda[0, 1]$. See [1, p. 454] and [2, p. 224]. The situation on $[a, 1]$ is quite different. In fact, for any $a > 0$ and $k \geqslant 0$, $d(x^k; \Lambda)$ actually decreases exponentially with n:

THEOREM 2. *Assume $\lambda_j \leqslant \beta j$, $j = 1, 2,..., n$, for some $\beta > 0$. Then*

$$d(x^k; \Lambda) \leqslant C_k \sqrt{n}(1 - a^{\beta/2})^n.$$

Proof. As in the previous proof, we begin with the inequality

$$d(x^k; \Lambda) \leqslant \left(\frac{1}{a}\right)^N \prod_{j=1}^{n} \left|\frac{\lambda_j - k}{\lambda_j + k + 2N}\right|.$$

Note then that if $\lambda_j \geqslant k$,

$$\left|\frac{\lambda_j - k}{\lambda_j + k + 2N}\right| \leqslant \frac{\lambda_j}{\lambda_j + 2N}$$

while if $\lambda_j < k$,

$$\left| \frac{\lambda_j - k}{\lambda_j + k + 2N} \right| \leqslant \frac{k}{k + 2N}.$$

In either case, however, since $x/(x + 2N)$ is an increasing function of $x > 0$,

$$\left| \frac{\lambda_j - k}{\lambda_j + k + 2N} \right| \leqslant \frac{\beta j}{\beta j + 2N}$$

except for finitely many j and thus

$$d(x^k; \Lambda) \leqslant C_k \left(\frac{1}{a}\right)^N \prod_{j=1}^{n} \frac{\beta j}{\beta j + 2N}.$$

We set $N = \beta \alpha n/2$ so that

$$d(x^k; \Lambda) \leqslant C_k \left(\frac{1}{a}\right)^{\beta \alpha n/2} \prod_{j=1}^{n} \frac{j}{j + \alpha n}.$$

Since

$$\prod_{j=1}^{n} \frac{j}{j + \alpha n} = \frac{\Gamma(n+1)\,\Gamma(\alpha n + 1)}{\Gamma(n + \alpha n + 1)},$$

we can apply Stirling's Formula to conclude (with perhaps a new constant C_k)

$$d(x^k; \Lambda) \leqslant C_k \sqrt{n} \left[\left(\frac{t\alpha}{1+\alpha}\right)^{\alpha} \frac{1}{1+\alpha} \right]^n,$$

where $t = (1/a)^{\beta/2}$. Thus choosing α so that $\alpha/(1+\alpha) = a^{\beta/2}$, we obtain Theorem 2.

EXAMPLE 1. If $a = \frac{1}{2}$, $\lambda_j \leqslant 2j$ then $d(x^k; \Lambda) \leqslant C_k \sqrt{n}/2^n$.

EXAMPLE 2. If $k = 0$, $\beta = 1$, Theorem 2 assures that the constant 1 can be approximated on $[a, 1]$ by a polynomial $P_\Lambda(x)$ (with $P_\Lambda(0) = 0$) to within $c\sqrt{n}(1 - \sqrt{a})^n$. If we take the special case $\lambda_j = j$, $j = 1, 2, \ldots, n$ the exact distance can be determined by noting that

$$\inf_{\{b_j\}} \left\| 1 - \sum_{j=1}^{n} b_j x^j \right\| = \left\| T_n\left(\frac{2x}{1-a} + \frac{a+1}{a-1}\right) \Big/ T_n\left(\frac{a+1}{a-1}\right) \right\|,$$

where T_n represents the nth degree Tchebychev polynomial on $[-1, 1]$, i.e., when $1 - \sum_{j-1}^{n} b_j x^j$ is a normalized translate of $T_n(x)$. Thus in this case

$$d(1; \Lambda) = 1 \Big/ T_n\left(\frac{a+1}{a-1}\right) \geqslant \left[\frac{1-a}{2(1+a)}\right]^n \geqslant \left(\frac{1-\sqrt{a}}{2}\right)^n.$$

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